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IFToMM Benchmark Problem
Four Bar Mechanism

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Chapter 1

Introduction

This report provides the results of the analysis of the IFToMM four bar linkage benchmark problem using the EoM software produced by the University of Windsor Vehicle Dynamics and Control research group. The problem consists of three uniform slender rods, of unit mass and length, with both gravity and a diagonal spring acting. The properties are summarized below. The problem is modified slightly by defining a torque on body three as the system input and the angular motion of body three as the system output. Note that there may be some round-off in the data describing the system properties. Please see the problem definition document for more precise values.

1.1 System Description

The properties of the bodies are given in Tables 1.1 and 1.2. The properties of the connecctions are given in Tables 1.3, 1.4, and 1.5.

Table 1.1: Body CG Locations and Mass

No.	Body Name	Location [m]	Mass [kg]
1	body 1	−0.308, 0.394, 0.000	1.000
2	body 2	−0.116, 0.788, 0.000	1.000
3	body 3	0.692, 0.394, 0.000	1.000

Table 1.2: Body Inertia Properties

No.	Body Name	Inertia [kg·m^2] (I_{xx}, I_{yy}, I_{zz}; I_{xy}, I_{yz}, I_{zx})
1	body 1	0.052, 0.032, 0.083; −0.040, 0.000, 0.000
2	body 2	0.000, 0.083, 0.083; 0.000, 0.000, 0.000
3	body 3	0.052, 0.032, 0.083; −0.040, 0.000, 0.000

Note: inertias are defined as the positive integral over the body, e.g., $I_{xy} = +\int r_x r_y \, dm$.

Table 1.3: Connection Location and Direction

No.	Connection Name	Location [m]	Unit Axis
1	pin 1	0.000, 0.000, 0.000	0.000, 0.000, 1.000
2	pin 2	−0.616, 0.788, 0.000	0.000, 0.000, 1.000
3	pin 3	0.384, 0.788, 0.000	0.000, 0.000, 1.000
4	pin 4	1.000, 0.000, 0.000	0.000, 0.000, 1.000

Table 1.4: Connection Locations

No.	Connection Name	Location [m]	Location [m]
1	spring 1	0.384, 0.788, 0.000	0.000, 0.000, 0.000

Table 1.5: Connection Properties

No.	Connection Name	Stiffness [N/m]	Damping [Ns/m]
1	spring 1	25	1

Chapter 2

Analysis

The EoM software automatically conducts a linear analysis after producing the linearized equations of motion. The results listed below show strong agreement with the IFToMM results.

2.1 Eigenvalue Analysis

The eigenvalue properties are given in Tables 2.1 and 2.2.

Table 2.1: Eigenvalues

No.	Real [rad/s]	Imaginary [rad/s]	Real [Hz]	Imaginary [Hz]
1	$-2.4238580931 \times 10^{-1}$	$2.1339550282 \times 10^{0}$	$-3.8576899686 \times 10^{-2}$	$3.3962949108 \times 10^{-1}$
2	$-2.4238580931 \times 10^{-1}$	$-2.1339550282 \times 10^{0}$	$-3.8576899686 \times 10^{-2}$	$-3.3962949108 \times 10^{-1}$

Note: oscillatory roots appear as complex conjugates.

Table 2.2: Eigenvalue Analysis

No.	Frequency (ω_n) [Hz]	Damping Ratio (ζ)	Time Constant (τ) [s]	Wavelength (λ) [s]
1	$3.4181335316 \times 10^{-1}$	$1.1285954551 \times 10^{-1}$	$4.1256540672 \times 10^{0}$	$2.9443850616 \times 10^{0}$
2	$3.4181335316 \times 10^{-1}$	$1.1285954551 \times 10^{-1}$	$4.1256540672 \times 10^{0}$	$2.9443850616 \times 10^{0}$

Notes: a) oscillatory roots are listed twice, b) negative time constants denote unstable roots.

There are 1 degrees of freedom.

There are 1 oscillatory modes, 1 damped modes, 0 unstable modes, and 0 rigid body modes.

2.2 Frequency Response Plots

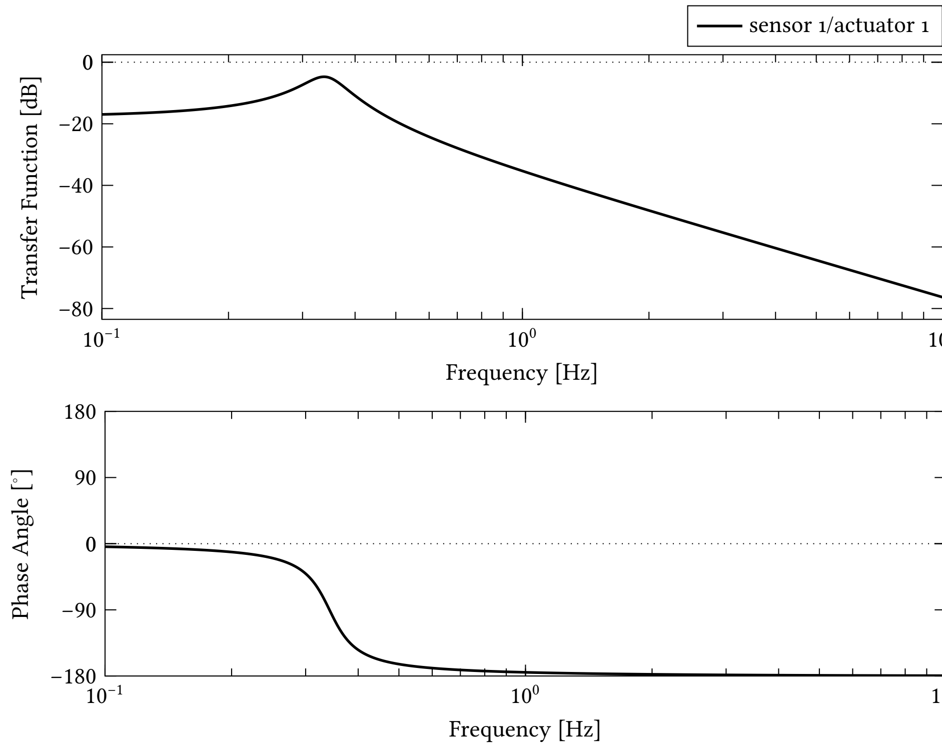

Figure 2.1: Frequency response: actuator 1

2.3 Steady State Gains

The steady state gains are given in Table 2.3.

Table 2.3: Steady State Gains

No.	Output/Input	Gain
1	sensor 1/actuator 1	$1.3008088000 \times 10^{-1}$

2.4 Equilibrium Analysis

The results of the equlibrium load analysis are given in Table 2.4.

Table 2.4: System Preloads

No.	Connector Name	Type	Load [N] or [Nm] (Components; Magnitude)
1	pin 1	force	-7.6693×10^{0}, 1.4715×10^{1}, 0.0000×10^{0}; 1.6594×10^{1}
2	pin 2	force	7.6693×10^{0}, -4.9050×10^{0}, 0.0000×10^{0}; 9.1037×10^{0}
3	pin 3	force	1.7778×10^{0}, -7.1791×10^{0}, 0.0000×10^{0}; 7.3959×10^{0}
4	pin 4	force	1.7778×10^{0}, 2.6309×10^{0}, 0.0000×10^{0}; 3.1753×10^{0}
5	spring 1	force	5.8915×10^{0}, 1.2084×10^{1}, 0.0000×10^{0}; -1.3444×10^{1}

Appendix A

Equations of Motion

The equations of motion are of the form

$$\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{M} & -\mathrm{G} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{w}} \\ \dot{\boldsymbol{u}} \end{Bmatrix} + \begin{bmatrix} \mathrm{V} & -\mathrm{I} & 0 \\ \mathrm{K} & \mathrm{L} & -\mathrm{F} \\ 0 & 0 & \mathrm{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{p} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathrm{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u} \end{Bmatrix}$$

The mass matrix of the system is

Row	Column	Value	Row	Column	Value
1	1	1.00000000×10^{0}	11	11	$8.33333333 \times 10^{-2}$
2	2	1.00000000×10^{0}	12	12	$8.33333333 \times 10^{-2}$
3	3	1.00000000×10^{0}	13	13	1.00000000×10^{0}
4	4	$5.17217120 \times 10^{-2}$	14	14	1.00000000×10^{0}
5	4	$4.04352220 \times 10^{-2}$	15	15	1.00000000×10^{0}
4	5	$4.04352220 \times 10^{-2}$	16	16	$5.17217120 \times 10^{-2}$
5	5	$3.16116213 \times 10^{-2}$	17	16	$4.04352220 \times 10^{-2}$
6	6	$8.33333333 \times 10^{-2}$	16	17	$4.04352220 \times 10^{-2}$
7	7	1.00000000×10^{0}	17	17	$3.16116213 \times 10^{-2}$
8	8	1.00000000×10^{0}	18	18	$8.33333333 \times 10^{-2}$
9	9	1.00000000×10^{0}	–	–	–

The damping matrix is

Row	Column	Value	Row	Column	Value
7	7	$1.92047302 \times 10^{-1}$	12	8	$4.03976349 \times 10^{-1}$
8	7	$3.93910061 \times 10^{-1}$	7	12	$1.96955030 \times 10^{-1}$
12	7	$1.96955030 \times 10^{-1}$	8	12	$4.03976349 \times 10^{-1}$
7	8	$3.93910061 \times 10^{-1}$	12	12	$2.01988174 \times 10^{-1}$
8	8	$8.07952698 \times 10^{-1}$	–	–	–

The stiffness matrix is

Row	Column	Value	Row	Column	Value
12	1	$-4.90500000 \times 10^{0}$	9	9	$-1.53386078 \times 10^{1}$
12	2	$-7.66930390 \times 10^{0}$	10	9	$-4.90500000 \times 10^{0}$
10	3	4.90500000×10^{0}	16	9	7.17906386×10^{0}
11	3	7.66930390×10^{0}	17	9	1.77782740×10^{0}
4	4	$-7.72851539 \times 10^{0}$	9	11	7.66930390×10^{0}
5	4	$-6.04203193 \times 10^{0}$	10	11	$-2.45250000 \times 10^{0}$
10	4	1.93212885×10^{0}	16	11	$-3.58953193 \times 10^{0}$
11	4	3.02101596×10^{0}	17	11	$-8.88913700 \times 10^{-1}$
4	5	$-6.04203193 \times 10^{0}$	7	12	7.94489172×10^{0}
5	5	$-4.72356565 \times 10^{0}$	8	12	8.62653960×10^{0}
10	5	1.51050798×10^{0}	12	12	8.14792175×10^{0}
11	5	2.36178282×10^{0}	18	12	$-8.88913700 \times 10^{-1}$
6	6	$-1.24520810 \times 10^{1}$	18	13	7.17906386×10^{0}
12	6	4.29391167×10^{0}	18	14	1.77782740×10^{0}
7	7	$-7.59168699 \times 10^{0}$	16	15	$-7.17906386 \times 10^{0}$
8	7	1.58897834×10^{1}	17	15	$-1.77782740 \times 10^{0}$
12	7	1.28498917×10^{1}	16	16	$-1.03635222 \times 10^{0}$
18	7	$-7.17906386 \times 10^{0}$	17	16	$7.00304100 \times 10^{-1}$
7	8	1.58897834×10^{1}	16	17	$-8.10203883 \times 10^{-1}$
8	8	1.72530792×10^{1}	17	17	$5.47486744 \times 10^{-1}$
12	8	1.62958435×10^{1}	18	18	$-4.88865472 \times 10^{-1}$
18	8	$-1.77782740 \times 10^{0}$	–	–	–

The velocity matrix is

Row	Column	Value	Row	Column	Value
1	1	0.00000000×10^{0}	–	–	–

The input force matrix is

Row	Column	Value	Row	Column	Value
18	1	1.00000000×10^{0}	–	–	–

The input force rate matrix is

Row	Column	Value	Row	Column	Value
1	1	0.00000000×10^{0}	–	–	–

The system is subject to constraints

$$\begin{bmatrix} \mathrm{J_h} & \mathbf{0} & \mathbf{0} \\ -\mathrm{J_h V} & \mathrm{J_h} & \mathbf{0} \\ \mathbf{0} & \mathrm{J_{nh}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} & \boldsymbol{p} \\ \dot{\boldsymbol{w}} & \boldsymbol{w} \\ \dot{\boldsymbol{u}} & \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Row	Column	Value	Row	Column	Value
1	1	1.00000000×10^{0}	18	19	1.00000000×10^{0}
6	1	1.00000000×10^{0}	23	19	1.00000000×10^{0}
2	2	1.00000000×10^{0}	19	20	1.00000000×10^{0}
7	2	1.00000000×10^{0}	24	20	1.00000000×10^{0}
3	3	1.00000000×10^{0}	20	21	1.00000000×10^{0}
8	3	1.00000000×10^{0}	25	21	1.00000000×10^{0}
3	4	$-3.93910061 \times 10^{-1}$	20	22	$-3.93910061 \times 10^{-1}$
5	4	$-1.00000000 \times 10^{0}$	22	22	$-1.00000000 \times 10^{0}$
8	4	$3.93910061 \times 10^{-1}$	25	22	$3.93910061 \times 10^{-1}$
10	4	$-1.00000000 \times 10^{0}$	27	22	$-1.00000000 \times 10^{0}$
3	5	$-3.07952698 \times 10^{-1}$	20	23	$-3.07952698 \times 10^{-1}$
4	5	1.00000000×10^{0}	21	23	1.00000000×10^{0}
8	5	$3.07952698 \times 10^{-1}$	25	23	$3.07952698 \times 10^{-1}$
9	5	1.00000000×10^{0}	26	23	1.00000000×10^{0}
1	6	$3.93910061 \times 10^{-1}$	18	24	$3.93910061 \times 10^{-1}$
2	6	$3.07952698 \times 10^{-1}$	19	24	$3.07952698 \times 10^{-1}$
6	6	$-3.93910061 \times 10^{-1}$	23	24	$-3.93910061 \times 10^{-1}$
7	6	$-3.07952698 \times 10^{-1}$	24	24	$-3.07952698 \times 10^{-1}$
6	7	$-1.00000000 \times 10^{0}$	23	25	$-1.00000000 \times 10^{0}$
11	7	1.00000000×10^{0}	28	25	1.00000000×10^{0}
7	8	$-1.00000000 \times 10^{0}$	24	26	$-1.00000000 \times 10^{0}$
12	8	1.00000000×10^{0}	29	26	1.00000000×10^{0}
8	9	$-1.00000000 \times 10^{0}$	25	27	$-1.00000000 \times 10^{0}$
13	9	1.00000000×10^{0}	30	27	1.00000000×10^{0}
10	10	1.00000000×10^{0}	27	28	1.00000000×10^{0}
15	10	$-1.00000000 \times 10^{0}$	32	28	$-1.00000000 \times 10^{0}$
8	11	$-5.00000000 \times 10^{-1}$	25	29	$-5.00000000 \times 10^{-1}$
9	11	$-1.00000000 \times 10^{0}$	26	29	$-1.00000000 \times 10^{0}$
13	11	$-5.00000000 \times 10^{-1}$	30	29	$-5.00000000 \times 10^{-1}$
14	11	1.00000000×10^{0}	31	29	1.00000000×10^{0}
7	12	$5.00000000 \times 10^{-1}$	24	30	$5.00000000 \times 10^{-1}$
12	12	$5.00000000 \times 10^{-1}$	29	30	$5.00000000 \times 10^{-1}$
11	13	$-1.00000000 \times 10^{0}$	28	31	$-1.00000000 \times 10^{0}$
17	13	$-1.00000000 \times 10^{0}$	34	31	$-1.00000000 \times 10^{0}$
12	14	$-1.00000000 \times 10^{0}$	29	32	$-1.00000000 \times 10^{0}$
16	14	1.00000000×10^{0}	33	32	1.00000000×10^{0}
13	15	$-1.00000000 \times 10^{0}$	30	33	$-1.00000000 \times 10^{0}$
13	16	$-3.93910061 \times 10^{-1}$	30	34	$-3.93910061 \times 10^{-1}$
15	16	1.00000000×10^{0}	32	34	1.00000000×10^{0}
13	17	$-3.07952698 \times 10^{-1}$	30	35	$-3.07952698 \times 10^{-1}$
14	17	$-1.00000000 \times 10^{0}$	31	35	$-1.00000000 \times 10^{0}$
11	18	$3.93910061 \times 10^{-1}$	28	36	$3.93910061 \times 10^{-1}$
12	18	$3.07952698 \times 10^{-1}$	29	36	$3.07952698 \times 10^{-1}$
16	18	$3.07952698 \times 10^{-1}$	33	36	$3.07952698 \times 10^{-1}$
17	18	$-3.93910061 \times 10^{-1}$	34	36	$-3.93910061 \times 10^{-1}$

The full state space equations:

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{Bmatrix}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \left[\begin{array}{ccc|c} -2.48083085 \times 10^{-2} & 9.95742968 \times 10^{-1} & 1.10417669 \times 10^{-2} & 0.00000000 \times 10^{0} \\ -2.20069272 \times 10^{0} & -2.06035319 \times 10^{-1} & 5.34408425 \times 10^{-1} & 0.00000000 \times 10^{0} \\ 0.00000000 \times 10^{0} & 0.00000000 \times 10^{0} & -1.00000000 \times 10^{0} & 1.00000000 \times 10^{0} \\ \hline 5.34408425 \times 10^{-1} & -1.10417669 \times 10^{-2} & 0.00000000 \times 10^{0} & 0.00000000 \times 10^{0} \end{array}\right]$$

$$\mathbf{E} = \begin{bmatrix} 9.99776479 \times 10^{-1} & -1.08181576 \times 10^{-2} & 0.00000000 \times 10^{0} \\ -1.08181576 \times 10^{-2} & 4.76413997 \times 10^{-1} & 0.00000000 \times 10^{0} \\ 0.00000000 \times 10^{0} & 0.00000000 \times 10^{0} & 0.00000000 \times 10^{0} \end{bmatrix}$$

The reduced state space equations:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \left[\begin{array}{cc|c} -7.48155790 \times 10^{-2} & 1.98305925 \times 10^{0} & 4.63754209 \times 10^{-2} \\ -2.31049266 \times 10^{0} & -4.09956040 \times 10^{-1} & 1.12225769 \times 10^{0} \\ \hline 2.67204213 \times 10^{-1} & -1.10417669 \times 10^{-2} & 0.00000000 \times 10^{0} \end{array}\right]$$